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Detection of singularities in wavelet and ridgelet analyses

By

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Abstract

We give two methods of detecting singularities in wavelet and ridgelet analyses. First, we give a characterization of the generalized two-microlocal Besov spaces in terms of the local Besov type conditions with dominating mixed smoothness. Secondly, we define microlocal ridgelet transforms after Candès and Donoho, and give the inversion formula for our ridgelet transforms.

§ 1. Introduction

The aim of this article is to give a characterization of the generalized two-microlocal Besov spaces in terms of wavelets and the inversion formula for the microlocal ridgelet transforms. In Section 2, we first recall two-microlocal Besov spaces and wavelets from Jaffard-Meyer [JM] and Moritoh-Yamada [MY]. We then give a characterization of the more generalized two-microlocal Besov spaces in terms of the local Besov type conditions with dominating mixed smoothness. We take account of an uncertainty function from Bony-Lerner [BL, Section 9.1] in the definition of the generalized two-microlocal Besov spaces. In Section 3, we define microlocal ridgelet transforms after Candès [C] and Candès-Donoho [CD], where the ridgelet transform is precisely the application of a one-dimensional wavelet transform to the slices of the Radon transform. If we use the wavelet transform defined in Moritoh [Mo1], we can detect directional singularities because of its microlocal properties. The main theorem of this section is the inversion formula for our microlocal ridgelet transforms.

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The results established in this article have already been given in the author's talks. They were invited by Real Analysis Section of Mathematical Society of Japan (at Ehime University in September, 2013 Autumn Meeting) and by RIMS Symposium on Several aspects of microlocal analysis (at Kyoto University in October, 2014). We also refer to the papers [Mo2] and [MT], where the details omitted in this article are to be found.

§ 2. Two-microlocal analysis

We first recall two-microlocal Besov spaces and wavelets from Jaffard-Meyer [JM] and Moritoh-Yamada [MY]. We then give a characterization of the more generalized two-microlocal Besov spaces in terms of the local Besov type conditions with dominating mixed smoothness. We take account of an uncertainty function from Bony-Lerner [BL, Section 9.1] in the definition of the generalized two-microlocal Besov spaces; the uncertainty function $1 + |\xi_1| + |x_2||\xi|$ $(x, \xi \in \mathbb{R}^2)$ corresponds to the uncertainty factor in Definition 2.6 below.

§ 2.1. Notation, definitions, and Moritoh-Yamada's result

Let $\mathbb{R}^n$ be n-dimensional real Euclidean space and $\mathbb{Z}^n$ be the lattice of all points $k = (k_1, \ldots, k_n) \in \mathbb{R}^n$, where the components $k_1, \ldots, k_n$ are integers. Let $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions on $\mathbb{R}^n$. If f belongs to the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, then

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx, \qquad \xi \in \mathbb{R}^n,$$

denotes the Fourier transform of f. Here $\langle x, \xi\rangle = \sum_{j=1}^n x_j \xi_j$ is the scalar product of $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$. The inverse Fourier transform $\mathcal{F}^{-1}g$ is given by

$$\mathcal{F}^{-1}g(x) = \check{g}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} g(\xi)d\xi, \qquad x \in \mathbb{R}^n.$$

The transforms $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended in the usual way from $\mathcal{S}$ to $\mathcal{S}'$.

Let $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \subset \mathcal{S}(\mathbb{R}^n)$ satisfy

1) supp $\varphi_j \subset \{x \in \mathbb{R}^n;\ 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}, \qquad j \in \mathbb{Z}$,

2) for every multi-index α there exists a positive number C_α such that

$$2^{j|\alpha|}|D^\alpha \varphi_j(x)| \leqslant C_\alpha, \qquad j \in \mathbb{Z},\ x \in \mathbb{R}^n,$$

and

3) $\sum_{j=-\infty}^{\infty} \varphi_j(x) \equiv 1, \qquad x \in \mathbb{R}^n \setminus \{0\}.$

Here D^α in 2) above are classical derivatives. Let $s > 0$ and $1 \leqslant p,\ q \leqslant \infty$. Then the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined as the set of all tempered distributions f (modulo polynomials) satisfying

$$\|f|\dot{B}^s_{p,q}(\mathbb{R}^n)\| = \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\mathcal{F}^{-1}(\varphi_j \mathcal{F} f)|L_p(\mathbb{R}^n)\|^q \right)^{1/q} < \infty$$

(usual modification if $q = \infty$). Here $\|\cdot|L_p(\mathbb{R}^n)\|$ stands for the usual L_p-norm. The definition of $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is independent of the choice $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$. See Triebel [T].

Let us now consider an orthonormal wavelet basis on $\mathbb{R}^n$. Such a basis is composed by translations and dilations of $2^n - 1$ functions $\psi^{(i)}(i \in \{0,1\}^n - (0,\ldots,0))$. We assume in the following that these wavelets are compactly supported smooth wavelets, whose supports are included in a ball centered at the origin. See Daubechies [D]. Let $\psi^{(i)}_{j,k}(x) = 2^{nj/2}\psi^{(i)}(2^j x - k),\ \ j \in \mathbb{Z},\ k \in \mathbb{Z}^n$. Then the wavelet decomposition of $f \in \mathcal{S}'$ will be written

$$f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} C_{j,k}\psi_{j,k}(x), \quad C_{j,k} = \langle f,\, \psi_{j,k}\rangle,$$

where we can forget the index i.

Let us recall the fact that $f \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ if and only if

$$\sum_{j\in\mathbb{Z}} 2^{jq(s+n/2-n/p)} \left(\sum_{k\in\mathbb{Z}^n} |C_{j,k}|^p \right)^{q/p} < \infty.$$

See Chapter VI, (10.5) in Meyer [M].

After these preliminaries we can define the local Besov spaces $B^s_{p,q}(U)$ and the two-microlocal Besov spaces $B^{s,s'}_{p,q}(x_0)$, where U is an open subset in $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. However, we treat only the case where $p = q$ in Theorem 2.3 below.

Definition 2.1. Let $s > 0$ and $1 \leqslant p,\ q \leqslant \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the local Besov space $B^s_{p,q}(U)$ if there exists an $F \in \dot{B}^s_{p,q}(\mathbb{R}^n)$ such that $f|_U = F|_U$, where $f|_U$ denotes the restriction of f to U. The norm $\|f|B^s_{p,q}(U)\|$ of f is then the infimum of all possible norms of F in $\dot{B}^s_{p,q}(\mathbb{R}^n)$.

Definition 2.2. Let $s > 0,\ s' \in \mathbb{R}$ and $1 \leqslant p,\ q \leqslant \infty$. Let $x_0 \in \mathbb{R}^n$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ is said to belong to the two-microlocal Besov space $B^{s,s'}_{p,q}(x_0)$ if the following two-microlocal estimate holds:

$$\|f|B^{s,s'}_{p,q}(x_0)\| = \left[\sum_{j\in\mathbb{Z}} 2^{jq(s+n/2-n/p)}\left\{\sum_{k\in\mathbb{Z}^n}\left|\left(1+2^j|k2^{-j}-x_0|\right)^{s'} C_{j,k}\right|^p\right\}^{q/p}\right]^{1/q} < \infty.$$

(usual modification if $p=\infty$ or $q=\infty$).

In order to state the local Besov type conditions in Theorems 2.3 and 2.7 below, we shall use the following notation: If $g(\rho)$ is a function of the real variable ρ, defined for all positive ρ, we write $g(\rho) = \mathcal{O}^{(p)}(\rho^{-s})$ if and only if

$$\int_0^R (g(\rho)\rho^s)^p \frac{d\rho}{\rho} = \int_0^R g(\rho)^p \rho^{sp-1} d\rho < \infty$$

for every positive number R. We can say that the symbol $\mathcal{O}^{(p)}$ is an L_p-version of the Hörmander symbol $\mathcal{O}^{(2)}$ used in Theorem 7.1 in [Hö].

Moritoh-Yamada's theorem is stated as follows:

Theorem 2.3. *Let $s>0$, $s'<0$, and $1 \leqslant p \leqslant \infty$. Let $x_0 \in \mathbb{R}^n$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^{s,s'}_{p,p}(x_0)$ if and only if there exists a decomposition $f = f_1 + f_2$ such that*

$$f_1 \in \dot{B}^s_{p,p}(\mathbb{R}^n),$$

and

$$\|f_2|B^{s+s'}_{p,p}(\{x\in\mathbb{R}^n; |x-x_0|<\rho\})\| = \mathcal{O}^{(p)}(\rho^{-s'}) \qquad \text{for every } \rho > 0.$$

Remark 2.4. Theorem 1.2 in [JM] treats the case where $p=\infty$ in this theorem.

Remark 2.5. Bony's two-microlocal space $H^{s,s'}_0(\mathbb{R}^n)$ is an L_2-Sobolev version of our general function space. See Definition 1.2 and Theorem 2.4 in [Bo]. Moreover, according to (2.17) of [Bo], we have the following fact: $u \in H^{s,-k}_0(\mathbb{R}^n)$, with k being a positive integer, if and only if $u = \sum_{|\alpha|\le k} x^\alpha u_\alpha$, where $u_\alpha \in H^{s-|\alpha|}(\mathbb{R}^n)$.

§ 2.2. The first main result

We treat only the case where $n=2$. Let us now consider an orthonormal wavelet basis on $\mathbb{R}^2$ composed by translations and dilations of $\psi(x_1)\psi(x_2)$, where $\psi(x)$ is a one-dimensional compactly supported smooth wavelet. Let $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$ for $j\in\mathbb{Z}, k\in\mathbb{Z}$. Then every $f\in\mathcal{S}'(\mathbb{R}^2)$ will be written

$$f(x) = \sum_{\boldsymbol{j}\in\mathbb{Z}^2}\sum_{\boldsymbol{k}\in\mathbb{Z}^2} C_{\boldsymbol{j},\boldsymbol{k}}\,\psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2),$$

where $\boldsymbol{j} = (j_1, j_2)$ and $\boldsymbol{k} = (k_1, k_2)$.

Let $\{\varphi_j(x)\}_{j=-\infty}^{\infty} \subset \mathcal{S}(\mathbb{R})$ satisfy the conditions 1) to 3) with $n = 1$ in Subsection 2.1. Let $s_1, s_2 > 0$ and $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. Then the homogeneous Besov space with dominating mixed smoothness $S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ is defined as the set of all tempered distributions f (modulo polynomials) satisfying

$$
\begin{aligned}
&\|f|S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)\| \\
&= \left[\sum_{j_2\in\mathbb{Z}} \left(\int_{\mathbb{R}} \left(\sum_{j_1\in\mathbb{Z}} \left(\int_{\mathbb{R}} \left| 2^{j_1 s_1 + j_2 s_2} \left(\varphi_{j_1}\varphi_{j_2}\hat{f}\right)^{\vee}(x_1, x_2)\right|^{p_1} dx_1\right)^{q_1/p_1}\right)^{p_2/q_1} dx_2\right)^{q_2/p_2}\right]^{1/q_2} < \infty,
\end{aligned}
$$

where $\boldsymbol{s} = (s_1, s_2), \boldsymbol{p} = (p_1, p_2), \boldsymbol{q} = (q_1, q_2)$, and

$$
(\varphi_{j_1}\varphi_{j_2}\hat{f})^{\vee}(x_1, x_2) = (\varphi_{j_1}(\xi_1)\varphi_{j_2}(\xi_2)\hat{f}(\xi_1, \xi_2))^{\vee}(x_1, x_2)
$$

(usual modification if at least one of p_1, p_2, q_1, q_2 is equal to ∞). This is the homogeneous counterpart of the space $SB^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$, whose definition is given in Definition 2 (ii) of Section 2.2.1 in Schmeisser-Triebel [ST]. The definition of $S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ is independent of the choice $\{\varphi_j(x)\}_{j=-\infty}^{\infty}$.

Let us recall the fact that $f \in S\dot{B}^{\boldsymbol{s}}_{\boldsymbol{p},\boldsymbol{q}}(\mathbb{R}^2)$ if and only if

$$
\sum_{j_2\in\mathbb{Z}} \left(\sum_{k_2\in\mathbb{Z}} \left(\sum_{j_1\in\mathbb{Z}} \left(\sum_{k_1\in\mathbb{Z}} |2^{j_1\tilde{s}_1 + j_2\tilde{s}_2} C_{\boldsymbol{j},\boldsymbol{k}}|^{p_1}\right)^{q_1/p_1}\right)^{p_2/q_1}\right)^{q_2/p_2} < \infty,
$$

where $\tilde{s}_i = s_i + 1/2 - 1/p_i$ $(i = 1, 2)$. See Bazarkhanov [B] and Vybiral [V]. We treat only the case where $\boldsymbol{p} = \boldsymbol{q} = (p, p) =: \boldsymbol{p}^*$, $1 \leq p \leq \infty$, for simplicity. We can define the local Besov space $SB^{\boldsymbol{s}}_p(\mathbb{R}_{x_1} \times A_\rho) := SB^{\boldsymbol{s}}_{\boldsymbol{p}^*,\boldsymbol{p}^*}(\mathbb{R}_{x_1} \times A_\rho)$ as in Definition 2.1, where $\mathbb{R}_{x_1} \times A_\rho$ denotes the horizontal strip $\{(x_1, x_2)\,;\, x_1 \in \mathbb{R}, |x_2| < \rho\}$ for $\rho > 0$. We can also give the definition of the two-microlocal Besov space with dominating mixed smoothness $SB^{(s_1,s_2),s_3}_p(\mathbb{R}_{x_1} \times \{0\})$ as in Definition 2.2; Bony-Lerner's uncertainty function $1 + |\xi_1| + |x_2||\xi|$ $(x, \xi \in \mathbb{R}^2)$ plays an important role. See [BL, Section 9.1].

Definition 2.6. Let $s_1, s_2 > 0$, $s_3 \in \mathbb{R}$, and $1 \leq p \leq \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^2)$ is said to belong to the two-microlocal Besov space with dominating mixed smoothness $SB^{(s_1,s_2),s_3}_p(\mathbb{R}_{x_1} \times \{0\})$ if the following two-microlocal estimate holds:

$$
\begin{aligned}
&\left\| f \,\middle|\, SB^{(s_1,s_2),s_3}_p(\mathbb{R}_{x_1} \times \{0\})\right\| \\
&:= \left[\sum_{\boldsymbol{j}\in\mathbb{Z}^2} \sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2 \\ k_2\neq 0}} 2^{(j_1\tilde{s}_1 + j_2\tilde{s}_2)p}(1 + 2^{j_1} + |k_2|2^{-j_2}2^{j_1\vee j_2})^{s_3 p}|C_{\boldsymbol{j},\boldsymbol{k}}|^p\right]^{1/p} < \infty,
\end{aligned}
$$

where $j_1 \vee j_2 = \max\{j_1, j_2\}$.

For positive numbers A and B, we write $A \lesssim B$ if $A \le cB$ holds for some positive constant c. Denoting by $F(U_x)$ some function space defined on $U_x \subset \mathbb{R}^2$ and by V_ξ some subset in $\mathbb{R}^2$, we write $f \in F(U_x,\, V_\xi)$ if the Fourier transform $\widehat{f}(\xi_1, \xi_2)$ of $f \in F(U_x)$ is supported in V_ξ.

Our main theorem of this section is the following:

Theorem 2.7. *Let $s_i > 0$, $s_3 < 0$, $s_i + s_3 > 0$ $(i = 1, 2)$, and $1 \le p \le \infty$. Then $f \in \mathcal{S}'(\mathbb{R}^2)$ belongs to $SB_p^{(s_1,s_2),s_3}(\mathbb{R}_{x_1} \times \{0\})$ if and only if there exists a decomposition $f = f_1 + f_2 + f_3 + f_4 + f_5$ such that*

$$f_1 \in S\dot{B}_p^{(s_1,s_2)}(\mathbb{R}^2), \quad f_2 \in S\dot{B}_p^{(s_1+s_3,s_2)}(\mathbb{R}^2),$$
$$f_3 \in S\dot{B}_p^{(s_1+s_3,s_2-s_3)}(\mathbb{R}^2,\, \{\xi \in \mathbb{R}^2; |\xi_1| \gtrsim |\xi_2|\}),$$

and for every $\rho > 0$

$$\big\| f_4 \,\big|\, SB_p^{(s_1+s_3,s_2)}(\mathbb{R}_{x_1} \times A_\rho,\, \{\xi \in \mathbb{R}^2; |\xi_1| \gtrsim |\xi_2|\})\big\| = \mathcal{O}^{(p)}(\rho^{-s_3}),$$
$$\big\| f_5 \,\big|\, SB_p^{(s_1,s_2+s_3)}(\mathbb{R}_{x_1} \times A_\rho,\, \{\xi \in \mathbb{R}^2; |\xi_1| \lesssim |\xi_2|\})\big\| = \mathcal{O}^{(p)}(\rho^{-s_3}).$$

Remark 2.8. The main idea of this theorem is that every f belonging to the generalized function space $SB_p^{(s_1,s_2),s_3}(\mathbb{R}_{x_1} \times \{0\})$ has a good decomposition $f = \sum_{i=1}^5 f_i$, where the terms f_4 and f_5 represent the singularities of the function f along the line $\mathbb{R}_{x_1}$; they satisfy the local Besov type conditions in the neighborhood of the x_1-axis. As we recalled in Subsection 2.1, every $f \in B_{p,p}^{s,s'}(x_0)$ has a good decomposition $f = f_1 + f_2$, where the term f_2 represents the singularities of the function f at the point x_0. The typical examples in Jaffard-Meyer [JM] are an indefinitely oscillating function of the form $x^\alpha \sin(1/x^\beta)$, and Riemann's nondifferentiable function $\sigma(x) = \sum_{n=1}^\infty (1/n^2)\sin(\pi n^2 x)$, where the Hölder regularity at a point x_0 depends on the Diophintine approximation properties of x_0. Multidimensional version of those singularities considered in [JM] will be studied in our forthcoming article by means of Theorem 2.7.

Remark 2.9. The two-microlocal Besov space of product type is introduced and characterized in [MT]. It is associated with the uncertainty functions $\lambda_i = 1 + |x_i||\xi_i|$ $(i = 1, 2)$; the norm of the wavelet coefficients $C_{\boldsymbol{j},\boldsymbol{k}}$ is defined by means of the weighted coefficients $2^{(j_1\tilde{s}_1 + j_2\tilde{s}_2)}(1 + |k_1|)^{s_1'}(1 + |k_2|)^{s_2'}|C_{\boldsymbol{j},\boldsymbol{k}}|$. We refer to the paper [MT] for further details.

§ 2.3. Outline of the proof of Theorem 2.7

We employ the method used in the proof of Theorem 2.3. We denote by C' the diameter of the support of the wavelet ψ. Let $f \in SB_p^{(s_1,s_2),s_3}(\mathbb{R}_{x_1} \times \{0\})$. Then its wavelet coefficients satisfy

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^2}\sum_{\substack{\boldsymbol{k}\in\mathbb{Z}^2\\ k_2\neq 0}} 2^{(j_1\tilde{s}_1+j_2\tilde{s}_2)p}(1+2^{j_1}+|k_2|2^{-j_2}2^{j_1\vee j_2})^{s_3 p}|C_{\boldsymbol{j},\boldsymbol{k}}|^p<\infty. \tag{2.1}$$

We write f as

$$f=\Bigg(\sum_{\substack{\operatorname{supp}\psi_{j_2,k_2}\ni 0\\ k_2\neq 0}}\sum\sum+\sum_{\substack{\operatorname{supp}\psi_{j_2,k_2}\not\ni 0\\ k_2\neq 0}}\sum\sum\Bigg)C_{\boldsymbol{j},\boldsymbol{k}}\psi_{j_1,k_1}(x_1)\psi_{j_2,k_2}(x_2)=:f_{\mathrm{I}}+f_{\mathrm{II}}.$$

If $\operatorname{supp}\psi_{j_2,k_2}\ni 0$, then $|k_2|$ is estimated from above by some constant comparable to C'. Therefore, if we decompose $f_{\mathrm{I}}=f_{\mathrm{I},1}+f_{\mathrm{I},2}+f_{\mathrm{I},3}$, where the three terms $f_{\mathrm{I},1}$, $f_{\mathrm{I},2}$ and $f_{\mathrm{I},3}$ correspond to the wavelet decompositions whose coefficients satisfy the three cases (I.1) $k_2\neq 0,\ j_1<0,\ j_2>j_1$, (I.2) $k_2\neq 0,\ j_1>0,\ j_2>0$ and (I.3) $k_2\neq 0,\ j_2<0,\ j_1>j_2$, respectively, then it follows from the conditions on j_1, j_2 and k_2, and from (2.1) that

$$f_{\mathrm{I},1}\in S\dot{B}_p^{(s_1,s_2)}(\mathbb{R}^2),\quad f_{\mathrm{I},2}\in S\dot{B}_p^{(s_1+s_3,s_2)}(\mathbb{R}^2),$$

$$f_{\mathrm{I},3}\in S\dot{B}_p^{(s_1+s_3,s_2-s_3)}(\mathbb{R}^2,\ \{\xi\in\mathbb{R}^2;|\xi_1|\gtrsim|\xi_2|\}).$$

Next we split the wavelet decomposition of f_{II} into three sums $f_{\mathrm{II}}=\sum_1+\sum_2+\sum_3$:

For a fixed positive number R, the first, $\sum_1$, corresponds to the wavelets whose supports do not intersect $\mathbb{R}_{x_1}\times A_R$, and we can forget this sum.

Next we consider the sum $\sum_2$ whose coefficients satisfy $2^{j_2}R\leqslant 10\,C'$; in that case, because $|k_2|=2^{j_2}|k_2|2^{-j_2}\lesssim 10C'$, we can treat the sum $\sum_2$ in the same way as f_{I}.

Finally we consider the remaining sum $\sum_3$ whose coefficients satisfy $2^{j_2}R\geqslant 10\,C'$. We decompose A_R into the dyadic strips as follows:

$$A_R=\bigcup_{m\in\mathbb{Z}:\,2^{-m}\le R}\{x=(x_1,x_2)\in\mathbb{R}^2\,;\,2^{-m-1}\le|x_2|\le 2^{-m}\}=:\bigcup_{m\in\mathbb{Z}:\,2^mR\ge 1}D_m. \tag{2.2}$$

By using this decomposition (2.2), we can write $\sum_3$ as $\sum_3=\sum_{\mathrm{i}}+\sum_{\mathrm{ii}}+\sum_{\mathrm{iii}}+\sum_{\mathrm{iv}}$, where the four terms $\sum_*=\sum_{*;j_1,j_2}\sum_{*;m}\sum_{*;k_1,k_2}$ $(*=\mathrm{i},\mathrm{ii},\mathrm{iii},\mathrm{iv})$ correspond to the wavelet decompositions whose coefficients satisfy the following four cases (i) to (iv), respectively:

$$\text{(i)}\quad j_1,j_2:\begin{cases}j_1<0,\\ 2^{j_2}R\ge 10C',\end{cases}\quad m:\begin{cases}2^mR\ge 1,\\ m>j_1,\ m>j_2,\end{cases}\quad k_1,k_2:\begin{cases}k_1\in\mathbb{Z},\\ k_2 2^{-j_2}\in D_m;\end{cases}$$

$$\text{(ii)}\quad j_1,j_2:\begin{cases}j_1>0,\\ 2^{j_2}R\ge 10C',\end{cases}\quad m:\begin{cases}2^mR\ge 1,\\ m>0,\ j_1+m>j_2,\end{cases}\quad k_1,k_2:\begin{cases}k_1\in\mathbb{Z},\\ k_2 2^{-j_2}\in D_m;\end{cases}$$

$$\text{(iii)}\quad j_1, j_2 : \begin{cases} j_1 > j_2, \\ 2^{j_2} R \geq 10C', \end{cases} m : \begin{cases} 2^m R \geq 1, \\ m < 0,\ m < j_1, \end{cases} \quad k_1, k_2 : \begin{cases} k_1 \in \mathbb{Z}, \\ k_2 2^{-j_2} \in D_m; \end{cases}$$

$$\text{(iv)}\quad j_1, j_2 : \begin{cases} j_2 > j_1, \\ 2^{j_2} R \geq 10C', \end{cases} m : \begin{cases} 2^m R \geq 1, \\ m < j_2,\ j_1 + m < j_2, \end{cases} k_1, k_2 : \begin{cases} k_1 \in \mathbb{Z}, \\ k_2 2^{-j_2} \in D_m. \end{cases}$$

First, from the conditions on j_1, j_2 and m of $\sum_{\mathrm{i}}$ and $\sum_{\mathrm{ii}}$, and from (2.1), it is easy to see that $\sum_{\mathrm{i}} \in S\dot{B}_p^{(s_1,s_2)}(\mathbb{R}^2)$ and $\sum_{\mathrm{ii}} \in S\dot{B}_p^{(s_1+s_3,s_2)}(\mathbb{R}^2)$, respectively. Next, because $\operatorname{supp} \psi_{j_2,k_2} \not\ni 0$, the terms corresponding to $m > j_2 + L(C')$ are negligible. Here $L(C')$ is a constant depending only on C'. Therefore from the conditions on j_1, j_2 and m of $\sum_{\mathrm{iii}}$, and from (2.1), we have

$$\sum_{\substack{j_1,j_2:\, 2^{j_2} R \geq 10C', \\ j_1 > j_2}} \sum_{\substack{m:\, 2^m R \geq 1, \\ m<0,\, m<j_1,\, m \leq j_2 + L(C')}} \sum_{\substack{k_2:\, k_2 2^{-j_2} \in D_m, \\ k_1 \in \mathbb{Z}}} 2^{(j_1 \tilde{s}_1 + j_2 \tilde{s}_2)p} 2^{(j_1 - m)s_3 p} |C_{\boldsymbol{j},\boldsymbol{k}}|^p < \infty,$$

that is,

$$\sum_{\substack{m:\, 2^m R \geq 1, \\ m<0}} 2^{-m s_3 p} \sum_{\substack{j_1,j_2:\, 2^{j_2} R \geq 10C', \\ j_1 > j_2,\, j_1 > m,\, j_2 \geq m - L(C')}} \sum_{\substack{k_2:\, k_2 2^{-j_2} \in D_m, \\ k_1 \in \mathbb{Z}}} 2^{j_1(\tilde{s}_1 + s_3)p} 2^{j_2 \tilde{s}_2 p} |C_{\boldsymbol{j},\boldsymbol{k}}|^p < \infty. \tag{2.3}$$

Therefore for $\rho < R$, and writing v instead of m in (2.3), we have

$$\sum_{\substack{v:\, 2^v R \geq 1, \\ v<0}} 2^{-v s_3 p} \sum_{\substack{j_1,j_2:\, 2^{j_2} \rho \geq 10C', \\ j_1 > j_2,\, j_1 > v,\, j_2 \geq v - L(C')}} \sum_{\substack{k_2:\, k_2 2^{-j_2} \in D_v, \\ k_1 \in \mathbb{Z}}} 2^{j_1(\tilde{s}_1 + s_3)p} 2^{j_2 \tilde{s}_2 p} |C_{\boldsymbol{j},\boldsymbol{k}}|^p < \infty.$$

Noting that $s_3 < 0$ and $2^{-v s_3 p} \simeq \sum_{u \leq v} 2^{-u s_3 p}$, we obtain

$$\sum_{\substack{u:\, 2^u R \geq 1, \\ u<0}} 2^{-u s_3 p} \sum_{\substack{j_1,j_2:\, 2^{j_2} \rho \geq 10C', \\ j_1 > j_2}} \sum_{\substack{v:\, u \leq v < 0, \\ v < j_1,\, v \leq j_2 + L(C')}} \sum_{\substack{k_2:\, k_2 2^{-j_2} \in D_v, \\ k_1 \in \mathbb{Z}}} 2^{j_1(\tilde{s}_1 + s_3)p} 2^{j_2 \tilde{s}_2 p} |C_{\boldsymbol{j},\boldsymbol{k}}|^p < \infty.$$

Consequently, we obtain the following integral representation for $f_4 := \sum_{\mathrm{iii}}$:

$$\int_0^R \Big(\rho^{s_3} \big\| f_4 \,|\, SB_p^{(s_1+s_3,s_2)}(\mathbb{R}_{x_1} \times A_\rho,\ \{\xi \in \mathbb{R}^2; |\xi_1| \gtrsim |\xi_2|\}) \big\| \Big)^p \frac{d\rho}{\rho} < \infty,$$

which means that the term f_4 satisfies the desired $\mathcal{O}^{(p)}$-condition. A similar treatment for the term $f_5 := \sum_{\mathrm{iv}}$ is possible.

Conversely since $s_3 < 0$, we have $(1 + 2^{j_1} + |k_2| 2^{-j_2} 2^{j_1 \vee j_2})^{s_3} \leq 1 \wedge 2^{j_1 s_3}$. Therefore, both $f_1 \in S\dot{B}_p^{(s_1,s_2)}(\mathbb{R}^2)$ and $f_2 \in S\dot{B}_p^{(s_1+s_3,s_2)}(\mathbb{R}^2)$ belong to our generalized two-microlocal Besov space. Let $f_3 \in S\dot{B}_p^{(s_1+s_3,s_2-s_3)}(\mathbb{R}^2,\ \{\xi \in \mathbb{R}^2; |\xi_1| \gtrsim |\xi_2|\})$. Then, since we have $2^{(j_1-j_2)s_3} = 2^{(j_1 \vee j_2 - j_2)s_3} \geq (1 + 2^{j_1} + |k_2| 2^{-j_2} 2^{j_1 \vee j_2})^{s_3}$ for $j_1 > j_2$ and $k_2 \neq 0$, we obtain that $f_3 \in SB_p^{(s_1,s_2),s_3}(\mathbb{R}_{x_1} \times \{0\})$.

The proof that the terms f_4 and f_5 belong to our generalized two-microlocal Besov space is omitted. It is similar to the proof of Theorem 2.3. See p. 282 of the paper [MY]. We also refer to the paper [MT].

§ 3. Microlocal ridgelet transforms and the inversion formula

We define microlocal ridgelet transforms after Candès [C] and Candès-Donoho [CD], where the ridgelet transform is precisely the application of a one-dimensional wavelet transform to the slices of the Radon transform. If we use the wavelet transform defined in Moritoh [Mo1], we can detect directional singularities because of its microlocal properties.

In Subsections 3.1 and 3.2, we recall the definitions and the inversion formulas for the Radon and wavelet transforms, respectively. After the definition of the microlocal ridgelet transforms is given in Subsection 3.3, the inversion formula for the microlocal ridgelet transforms is stated with the outline of the proof in Subsection 3.4.

§ 3.1. Radon transforms

We first follow Helgason [H]. Let f be a compactly supported, smooth function on $\mathbb{R}^n$, where we assume $n \geq 2$. For $\omega \in S^{n-1}$ and $p \in \mathbb{R}$, the hyperplane $L(\omega, p)$ is defined as the set $\{x \in \mathbb{R}^n;\ \langle x, \omega \rangle = p\}$. Here S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. Note that $L(\omega, p) = L(-\omega, -p)$. The collection of all hyperplanes is denoted by $\mathbb{P}^n$, being furnished with the obvious topology. Note that the functions on $\mathbb{P}^n$ are identified with the functions φ on $S^{n-1} \times \mathbb{R}$ which are even: $\varphi(\omega, p) = \varphi(-\omega, -p)$. The Radon transform $Rf(\omega, p)$ is defined as follows:

$$\mathrm{R}f(\omega, p) := \int_{L(\omega,p)} f(x)\, dx,$$

where dx denotes the $(n-1)$-dimensional Lebesgue measure on $L(\omega, p)$. For a compactly supported, smooth function $g(\omega, p)$ on $\mathbb{P}^n$, the dual Radon transform $\mathrm{R}^* g(x)$ is defined as follows:

$$\mathrm{R}^* g(x) := \int_{S^{n-1}} g(\omega, \langle x, \omega \rangle)\, d\omega,$$

where $d\omega$ denotes the area element on the unit sphere S^{n-1}. We denote by $\hat{g}(\omega, \hat{p})$ the one-dimensional Fourier transform of $g(\omega, p)$ with respect to p, and define the operator Λ by

$$(\Lambda g)^\wedge(\omega, \hat{p}) = |\hat{p}|^{n-1} \hat{g}(\omega, \hat{p}).$$

Then we have the following inversion formula (3.1):

$$f(x) = C_R^{-1}\, \mathrm{R}^* \Lambda \mathrm{R} f(x), \tag{3.1}$$

where the constant $C_R = 2^n\pi^{n-1}$. If we denote by $\hat{f}(\xi)$ the n-dimensional Fourier transform of $f(x)$, then we have the following formula (the projection-slice theorem):

$$(\mathrm{R}f)^\wedge(\omega,\hat{p}) = \hat{f}(\hat{p}\,\omega).$$

The Radon transform can be defined for distributions in $\mathcal{E}'(\mathbb{R}^n)$, the space of distributions of compact support, and the inversion formula can be extended to distributions. See Theorem 5.5 of Chapter I in [H].

§ 3.2. Wavelet transforms

We recall our wavelet transforms from [Mo1]. Our wavelet function $\psi(x)$ on $\mathbb{R}^n$ satisfies the following conditions (i) and (ii):

(i) $\hat{\psi}(\xi)$ has a compact support containing $(0,\dots,0,1)$ in the interior, and not containing the origin;

(ii) $\psi(x) = \psi(\rho x)$ for every rotation ρ with $\rho(0,\dots,0,1) = (0,\dots,0,1)$.

For $\xi \in \mathbb{R}^n - \{0\}$, let ρ_ξ be any rotation satisfying $\rho_\xi(\xi/|\xi|) = (0,\dots,0,1)$, and put $\psi_\xi(x) := |\xi|^n\psi(|\xi|\rho_\xi x)$, or $\hat{\psi}_\xi(\hat{x}) := \hat{\psi}(|\xi|^{-1}\rho_\xi\hat{x})$. Then our wavelet transform $W_\psi f(x,\xi)$ is defined as follows:

$$W_\psi f(x,\xi) = \int_{\mathbb{R}^n} f(t)\overline{\psi_\xi(t-x)}\,dt, \qquad (x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n - \{0\}).$$

The inversion formula for our wavelet transforms can be stated as follows:

$$\begin{aligned} f(x) &= C_\psi^{-1} \iint_{\mathbb{R}^n\times\mathbb{R}^n} W_\psi f(t,\xi)\psi_\xi(x-t)\,dt\,d\xi/|\xi|^n \\ &= C_\psi^{-1} \int_{\mathbb{R}^n} f * \widetilde{\psi_\xi} * \psi_\xi(x)\,d\xi/|\xi|^n, \end{aligned} \tag{3.2}$$

where $\widetilde{\psi_\xi}(x) := \overline{\psi_\xi(-x)}$, and $C_\psi := (2\pi)^n \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 d\xi/|\xi|^n$. Note that this formula (3.2) gives a microlocalization of the Calderón formula. See also Theorem 1.2 of Chapter 1 in Frazier-Jawerth-Weiss [FJW].

§ 3.3. Definition: Microlocal ridgelet transforms

After the preliminaries in Subsections 3.1 and 3.2, we can now define our microlocal ridgelet transform. By using the wavelet ψ, we first define the ridgelet function $\varphi(\omega,p)$ on $S^{n-1}\times\mathbb{R}$ as follows:

$$\varphi(\omega,p) := \Lambda^{1/2}(\mathrm{R}\psi)(\omega,p), \text{ or } \hat{\varphi}(\omega,\hat{p}) = |\hat{p}|^{(n-1)/2}\hat{\psi}(\hat{p}\,\omega). \tag{3.3}$$

For $\xi \in \mathbb{R}^n - \{0\}$, put

$$\varphi_\xi(\omega, p) = |\xi|^{(n+1)/2} \varphi(\rho_\xi \omega, |\xi| p), \text{ or } \hat{\varphi}_\xi(\omega, \hat{p}) = |\hat{p}|^{(n-1)/2} \hat{\psi}_\xi(\hat{p}\, \omega).$$

Let f be a compactly supported, smooth function on $\mathbb{R}^n$, where we assume $n \geq 2$. Then the microlocal ridgelet transform of f is defined as follows:

$$\mathcal{R}_\varphi f(\omega, p; \xi) := \int_{\mathbb{R}^n} f(x) \overline{\varphi_\xi(\omega, \langle x, \omega \rangle - p)}\, dx.$$

We have an equivalent representation:

$$\mathcal{R}_\varphi f(\omega, p; \xi) = \int_{\mathbb{R}} \mathrm{R} f(\omega, q) \overline{\varphi_\xi(\omega, q - p)}\, dq.$$

Remark 3.1. Fix a vector $\xi \in \mathbb{R}^n - \{0\}$. Then the microlocal ridgelet transform $\mathcal{R}_\varphi f(\omega, p; \xi)$ has its support with respect to ω in a neighborhood of the direction $\xi/|\xi|$, and represents the Radon data of $f(x)$ in a neighborhood of $L(\omega, p)$. These observations explain why our ridgelet transform $\mathcal{R}_\varphi f(\omega, p; \xi)$ can be said to be a microlocalization of the Candès ridgelet transform.

§ 3.4. The second main result: Inversion formula

Our inversion formula, which is a microlocal analogue of the formula for the Candès ridgelet transform, reads as follows:

Theorem 3.2. *The function f, which is smooth and compactly supported on $\mathbb{R}^n$, can be recovered from the microlocal ridgelet transform by means of the inversion formula:*

$$f(x) = C^{-1} \int_{\mathbb{R}^n} \left[\int_{S^{n-1}} \int_{\mathbb{R}} \mathcal{R}_\varphi f(\omega, p; \xi) \varphi_\xi(\omega, \langle x, \omega \rangle - p)\, dp\, d\omega \right] d\xi / |\xi|^n,$$

where the constant $C := C_R C_\psi = 2^{2n} \pi^{2n-1} \int_{\mathbb{R}^n} |\hat{\psi}(\xi)|^2 d\xi / |\xi|^n$.

Outline of the proof. Apply the Radon transform to (3.2). Then the projection-slice theorem yields

$$\mathrm{R} f(\omega, p) = C_\psi^{-1} \int_{\mathbb{R}^n} \mathrm{R} f(\omega, \cdot) * \mathrm{R}\widetilde{\psi_\xi}(\omega, \cdot) * \mathrm{R}\psi_\xi(\omega, \cdot)(p)\, d\xi / |\xi|^n. \tag{3.4}$$

Apply the pseudodifferential operator Λ to (3.4) to obtain

$$\Lambda \mathrm{R} f(\omega, p) = C_\psi^{-1} \int_{\mathbb{R}^n} \mathrm{R} f(\omega, \cdot) * \widetilde{\varphi_\xi}(\omega, \cdot) * \varphi_\xi(\omega, \cdot)(p)\, d\xi / |\xi|^n, \tag{3.5}$$

where $\widetilde{\varphi_\xi}(\omega,p)=\overline{\varphi_\xi(\omega,-p)}$. Finally, apply the dual Radon transform to (3.5) to obtain

$$\mathrm{R}^*\Lambda\mathrm{R}f(x)=C_\psi^{-1}\int_{\mathbb{R}^n}\left[\int_{S^{n-1}}\int_{\mathbb{R}}\mathcal{R}_\varphi f(\omega,p;\xi)\varphi_\xi(\omega,\langle x,\omega\rangle-p)\,dp\,d\omega\right]d\xi/|\xi|^n,$$

that is,

$$f(x)=C_R^{-1}C_\psi^{-1}\int_{\mathbb{R}^n}\left[\int_{S^{n-1}}\int_{\mathbb{R}}\mathcal{R}_\varphi f(\omega,p;\xi)\varphi_\xi(\omega,\langle x,\omega\rangle-p)\,dp\,d\omega\right]d\xi/|\xi|^n.$$

□

A microlocalization of the Candès ridgelet spaces is now given. Let $s\geq 0$ and $1\leq u,\ v\leq\infty$. Then, for every $\xi\in\mathbb{R}^n-\{0\}$, the microlocal ridgelet space $\dot{\mathcal{R}}^s_{u,v}(\xi/|\xi|)$ is defined as follows: A function f is said to belong to the microlocal ridgelet space $\dot{\mathcal{R}}^s_{u,v}(\xi/|\xi|)$ if there exists a ridgelet φ with (3.3) such that the following estimate holds:

$$\left[\int_0^\infty\left(\int_{S^{n-1}\times\mathbb{R}}\left||\xi|^s\mathcal{R}_\varphi f(\omega,p;\xi)\right|^u d\omega\,dp\right)^{v/u}d|\xi|/|\xi|\right]^{1/v}<\infty.$$

A further analysis by means of this microlocal space is omitted here. See [Mo2].

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